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The Electric and Magnetic State of the Interior of the Earth, as Inferred from Terrestrial Magnetic Variations

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X. *The Electric and Magnetic State of the Interior of the Earth, as inferred from Terrestrial Magnetic Variations.*

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1. INTRODUCTION.

1.1. By spherical harmonic analysis it is possible, as was shown by GAUSS,[1] to distinguish between the parts of the earth's magnetic field, at any time, which originate respectively within and above the earth's surface. In this way GAUSS confirmed and extended GILBERT'S conclusion that the field is almost entirely of internal origin.

1.2. Sir ARTHUR SCHUSTER[2] applied the method to the field of the daily magnetic variation at the earth's surface, and found that the major part is of external origin, but that there is also a part produced within the earth. He attributed the latter to electric currents induced in the earth by the outer varying field, which he regarded as primary. In co-operation with Prof. H. LAMB he showed that the relation between the two parts of the field is consistent with this hypothesis; the calculations referred to currents induced in a uniformly conducting sphere. They showed that the conducting sphere must be distinctly smaller than the earth, that is, it is an inner core, and not the whole earth, which is effective.

1.3. Definite estimates both of the size and the conductivity (κ) of the core were made by S. CHAPMAN.[3] The estimates of (*a*) the thickness of the non-conducting shell surrounding the core, and (*b*) of κ, were (*a*) about 250 km., and (*b*) $3 \cdot 6.10^{-13}$ C.G.S. units. This conductivity is similar to that of moist earth, and distinctly less than that of sea water (4.10^{-11}).

1.4. In these calculations the magnetic permeability μ was assumed to be unity, and the influence of the oceans, and of the water-bearing strata near the earth's surface, was ignored. S. CHAPMAN and T. T. WHITEHEAD[4] showed that the periodic magnetic variations suffice only to indicate the value of the ratio κ/μ, and cannot determine κ and μ separately. They found that the induced currents are scarcely affected by any probable depth of moist earth, but that a comparatively shallow oceanic shell (*e.g.*, half a mile deep) *over the whole earth* would induce a field of intensity comparable with

[1] GAUSS, "Allgemeine Theorie der Erdmagnetismus," p. 13.

[2] SCHUSTER. 'Phil. Trans.,' A, vol. 180, p. 467 (1889); with appendix by LAMB.

[3] CHAPMAN, 'Phil. Trans.,' A, vol. 218, p. 1 (1919).

[4] CHAPMAN and WHITEHEAD, 'Trans. Camb. Phil. Soc.,' vol. 22, p. 463 (1922).

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that due to the supposed core; the actual oceans are likely to have an appreciable effect on the induced field, though their irregular and ill-connected form reduces their influence.

1.5. CHAPMAN[5] analyzed the average field of magnetic storms, and showed that it originates mainly above the earth, but has also a minor part of internal origin[6]: as in the case of the daily magnetic variations, this is naturally interpreted as induced by the primary outer part of the field. CHAPMAN and WHITEHEAD (*loc. cit.*) considered the relation between the outer and inner parts of the main axially-symmetrical component of the field, adopting the hypothesis of induction, in a core of which the conductivity was of the same order (between 10^{-13} and 4.10^{-13}) as that inferred from the daily magnetic variations. This part of their calculations was somewhat rough and provisional, and, as will appear later, the effective conductivity which we now infer is considerably higher.

1.6. Terrestrial magnetic phenomena can thus afford indications as to the properties of the earth's interior, which can hardly be obtained in any other way. Probably the most interesting and important geophysical information derived from the magnetic data will be gained from the main field and its secular variation, when at last it becomes possible to interpret them satisfactorily. Meanwhile it seems desirable to proceed further with the study of the induction effects associated with the daily magnetic variations and magnetic disturbance; they present several important problems which require solution.

1.7. In view of the very large part of the globe which is covered by water, it is desirable to examine further the extent to which the oceans shield the core from the varying outer field. This can be done by considering one or more limited oceans of simple form, bounded, say, by meridians across which current cannot flow.[7]

1.8. It is desirable also to make a detailed examination of the induced part of the field of magnetic storms, to determine whether it is really consistent with the properties of the conducting core as deduced from the daily magnetic variations. It is hardly to be expected that this will be the case, because the supposed uniformly conducting core can only be regarded as a convenient mathematical model consistent with certain observed facts; it may suffice, with certain values of κ and the radius, for one type of magnetic variation, while for another type different values may be necessary. If so, the difference between the two sets of values may indicate how the actual core differs from the simple model.

1.9. This question forms the main subject of the present paper, which is a further development of that by CHAPMAN and WHITEHEAD, already referred to. Mathematically all such discussions are based upon LAMB's work on electromagnetic induction in spheres[8]; the theory for a uniform conducting sphere is complete as regards induction

[5] CHAPMAN, 'Proc. Roy. Soc.,' A, vol. 95, p. 61 (1918).
[6] CHAPMAN, 'M.N.R.A.S.,' vol. 79, p. 70 (1918).
[7] CHAPMAN, 'J. Lond. Math. Soc.,' vol. 2, p. 131 (1927).
[8] LAMB, 'Phil. Trans.,' vol. 174, p. 526 (1883).

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by periodic magnetic fields, but in the course of our work a further development of the theory for *aperiodic* fields has proved necessary; the extensions are given elsewhere in two papers by A. T. PRICE.[9] The distinction between the cases of periodic and aperiodic fields is somewhat analogous to that between (*a*) the forced vibrations of an elastic dissipative system, due to external forces of assigned period, and (*b*) the motion of the system from rest, due to external aperiodic forces; in the latter case, the modes of free vibration have to be found, and their amplitude determined to satisfy the initial conditions.

1.10. Seismological evidence[10] indicates a fairly sharp change in the constitution of the earth at a depth of about 15 km., but there is no independent indication of a change of properties near the level, 250 km. deep, where the conducting core begins (according to CHAPMAN'S estimate based on the daily magnetic variations). There may, in fact, be no sudden increase of κ at this level, but only a gradual increase, and this increase may continue to much higher values[11] of κ than $3\cdot6.10^{-13}$. To throw light on this point we here examine the actual distribution of the induced currents in the core (on the supposition that it is uniformly conducting and that $\kappa = 3\cdot65.10^{-13}$), both for the daily variations and for magnetic storms. It is found that the currents penetrate further in the latter case, but that below about 2000 km. they are small and have only a slight influence on the surface magnetic field. Thus the region about which these magnetic variations can afford information is unlikely to extend below this depth and may fall far short of it if the conductivity increases downwards. The comparison of the induction-effects for the daily variations and for magnetic storms does, in fact, indicate that the conductivity is not uniform below 250 km., but continues to increase, and at a quite rapid rate. By considering the magnetic variations over a wider range of period (or, in the case of aperiodic variations, of time scale), it may prove feasible to determine the radial distribution of conductivity for some distance below 200 km., though the data now available scarcely suffice for this purpose. The investigation will require a difficult extension of the mathematical theory if the conductivity is supposed to vary continuously; but it may prove sufficient to treat the earth as a series of concentric shells each of uniform conductivity; the formulæ required in this case have been given by CHAPMAN and WHITEHEAD (*loc. cit.*). In the present paper the detailed analysis is concerned solely with a uniformly conducting core.

The slower the magnetic variations, the deeper do their induced currents penetrate; hence the variations of longest period must be looked to for information about the deepest levels. The slowest known variation of external origin is the annual variation; this is very small and difficult to determine accurately, but in the future it may become possible to apply it to the present purpose.

[9] PRICE, 'Proc. Lond. Math. Soc.'; ser. 2, vol. 31, p. 217 (1930), and a second paper in preparation.

[10] JEFFREYS, "The Earth," 2nd ed. (1929), p. 159.

[11] CHAPMAN, 'Nature,' vol. 124, p. 24 (1929).

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1.11. In conclusion, some errors and misconceptions in two papers[12, 13] bearing on electromagnetic induction within the earth by varying external magnetic fields are discussed (§§ 9, 10).

2. The Magnetic Data, and their Harmonic Analysis.

2.1. The potential V of the magnetic field at and near the surface of the earth may at any instant be expressed as a series of spherical harmonic terms

$$V = \sum_n \sum_p V_n^p, \tag{1}$$

where V_n^p depends on θ the north polar distance (or colatitude), and on λ the east longitude, solely through the surface-harmonic factor

$$P_n^p (\cos \theta) (A \cos p\lambda + B \sin p\lambda) ; \tag{2}$$

$P_n^p (\cos \theta)$ denotes the associated Legendre function of degree n and order p ; n and p are positive integers or zero, and $p \leq n$; A and B are functions of r, of the type indicated in the following general expression for V_n^p :—

$$V_n^p = \left\{\left(E^p_{n,a} \frac{r^n}{a^{n-1}} + I^p_{n,a} \frac{a^{n+2}}{r^{n+1}}\right) \cos p\lambda + \left(E^p_{n,b} \frac{r^n}{a^{n-1}} + I^p_{n,b} \frac{a^{n+2}}{r^{n+1}}\right) \sin p\lambda\right\} P_n^p (\cos \theta). \tag{3}$$

The terms in the potential which are of positive degree in r relate to the part of the field that has its origin above the earth's surface, while the other terms are associated with the part that originates within the earth: the factors E_n^p, I_n^p associated with these terms are chosen so as to suggest this external and internal character. The factors E, I are, in general, functions of the time.

2.2. Each of them can be analyzed into a constant mean part, corresponding to the earth's permanent main field, together with a variable part, which contains some periodic terms, and some which are irregular. The former include the solar and lunar diurnal terms, and the latter the terms corresponding to magnetic storms: these alone are discussed in this paper. There are, in addition, the slowly varying terms corresponding to the annual and secular variations, and some terms due to small and rapid pulsations of the field.

2.3. Any periodic component of V_n^p, associated with time factors $\cos \alpha t$ or $\sin \alpha t$, can conveniently be represented by the real part of the sum of a number of expressions of the type

$$\left(E_n^{p,\alpha} \frac{r^n}{a^{n-1}} + I_n^{p,\alpha} \frac{a^{n+2}}{r^{n+1}}\right) e^{i(p\lambda \pm \alpha t)} P_n^p (\cos \theta), \tag{4}$$

where $E_n^{p,\alpha}$ and $I_n^{p,\alpha}$ are complex numbers, called amplitude factors. The modulus

[12] Chapman and Whitehead, 'Proc. Toronto Int. Math. Congress,' 1928, p. 313.

[13] Maris and Hulburt, 'Phys. Rev.,' vol. 33, p. 412 (1929).

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of the complex ratio $E_n^{p,\alpha}/I_n^{p,\alpha}$ is called the amplitude ratio of the external to the internal part of this component of the potential term V_n^p, while the argument of this ratio is called the phase difference between them.

2.4. The harmonics V_n^p are determined at any instant by the geographical distribution of the three magnetic elements at that instant. By making the analysis for a series of times, the factors $E^p_{n,a}$, $I^p_{n,a}$, $E^p_{n,b}$, $I^p_{n,b}$ (for any desired values of n and p), could be found as functions of the time. In practice, however, it proves more convenient to make the time-analysis first, for the elements at the separate observatories. The periodic components are expressed as FOURIER's series, and the geographical distribution of the coefficients in these series determines the corresponding periodic components of the various harmonic terms V_n^p. This has been done for both the solar[2, 3] and lunar[3] diurnal variations; the principal harmonics are those for which $p = 1, 2, 3$, or 4, and $n = p$ or $n = p + 1$. Moreover, they are found to depend mainly on local time; that is to say, when t is reckoned at the rate 2π per (respectively solar or lunar) day,

$$\alpha = p, \tag{5}$$

while if t is measured in seconds,

$$\alpha = 2\pi p/86400 \tag{5a}$$

in the case of the solar day.

2.5. The additional variations present during magnetic storms have been discussed by N. A. F. MOOS[14], S. CHAPMAN[5] and G. ANGENHEISTER[15]; they may be analyzed into the storm-time variation, the disturbance diurnal variations, and an irregular part. The first of these, which is alone considered in this paper, represents the part of the variation that is symmetrical about the earth's axis; it depends only on latitude and storm-time (that is, time measured from the commencement of the storm). Since this part is independent of the longitude, its potential will contain only zonal harmonics ($p = 0$). The observations clearly indicate that the most important harmonic term is the first (proportional to P_1 (cos θ)), and that higher harmonics are relatively small[6]; an actual determination of them, as functions of the time, is made here for the first time, on the basis of the graphs[5] of the storm-time changes of the north and vertical magnetic force (X and Z),[16] reproduced in fig. 5; these graphs refer to the average (north) magnetic latitudes 22°, 40°, 53°. It may be safely assumed that the storm-time field is nearly

[14] MOOS, "Colaba Magnetic Observations 1846–1905," vol. 2, ch. 10 (1910).

[15] ANGENHEISTER, 'Göttingen Nach.,' Math. Phys., 1924, p. 1.

[16] A zonal harmonic in the potential V can give rise only to north (or south) and vertical components of magnetic force, *i.e.*, there can be no corresponding east or west component. The analysis[5] showed that the only large storm-time changes are those in H (horizontal force, north or nearly north at the stations considered), though those in V, while small, are definite. The storm-time changes in declination, or east force, were small and irregular; they are probably accidental, due to the limited number and individual differences of the storms analysed.

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symmetrical with respect to the equator, so that only harmonics of odd degree need consideration. The data suffice to determine three of these, P_1, P_3, P_5; the others are supposed to be negligible in comparison with them.

Thus, let

(6) $$\Omega = \Sigma \left(e_n \frac{r^n}{a^{n-1}} + i_n \frac{a^{n+2}}{r^{n+1}} \right) P_n (\cos \theta), \quad n = 1, 3, 5,$$

where Ω denotes the storm-time part of V, while e_n and i_n are functions of the time. Since

(7) $$X = \left(\frac{\partial \Omega}{r\, \partial \theta} \right)_{r=a}, \qquad Z = \left(\frac{\partial \Omega}{\partial r} \right)_{r=a},$$

simultaneous values of X and Z in three latitudes suffice to determine e_n and i_n for $n = 1, 3, 5$. The values found for various epochs after the commencement of the storm are shown in Table I. At the earth's surface ($r = a$), $\Omega = \Sigma\, (e_n + i_n)\, P_n (\cos \theta)$; the coefficients $e_n + i_n$ of the three surface harmonics are also given in Table I, and are illustrated, as functions of the time, in fig. 1. Fig. 2 shows the time-variations of e_1 and i_1 separately.

Table I.—Values of the coefficients in the expression

$$\Omega = \{e_1 r + i_1 (a^3/r^2)\}\, P_1 + \{e_3 (r^3/a^2) + i_3 (a^5/r^4)\}\, P_3 + \{e_5 (r^5/a^4) + i_5 (a^7/r^6)\}\, P_5,$$

representing the observed H.F. and V.F. "storm-time" changes.

Unit = 1γ. Time measured from half-hour before commencement of storm.

	0 hour.	1 hour.	3 hours.	6 hours.	12 hours.	18 hours.	24 hours.	30 hours.	36 hours.	42 hours.	48 hours.
$e_1 + i_1$	0	− 16	− 8	14	36	38	37	33	31	28	27
e_1	0	− 11	− 6	11	26	28	26	24	23	21	20
i_1	0	− 5	− 2	3	10	10	11	9	8	7	7
$e_3 + i_3$	0	− 1	0	1	2	0	−1	1	0	1	0
e_3	0	− 0·5	1	2	1	0	−2	0	−1	0	−1
i_3	0	− 0·5	− 1	−1	1	0	−1	1	1	1	1
$e_5 + i_5$	0	− 1	− 4	−4	− 1	−1	− 0·5	−2	−1	−2	−1
e_5	0	0	− 1·5	−1	0	0	− 0·5	−1	−1	−1	− 0·5
i_5	0	− 1	− 2·5	−3	− 1	−1	0	−1	0	−1	− 0·5

2.6. It is clear that, except during the first ten hours of the storm, the field is adequately represented by the single harmonic P_1. The values of P_3 and P_5, while scarcely likely to be accurate (because of the limited data on which they are based),

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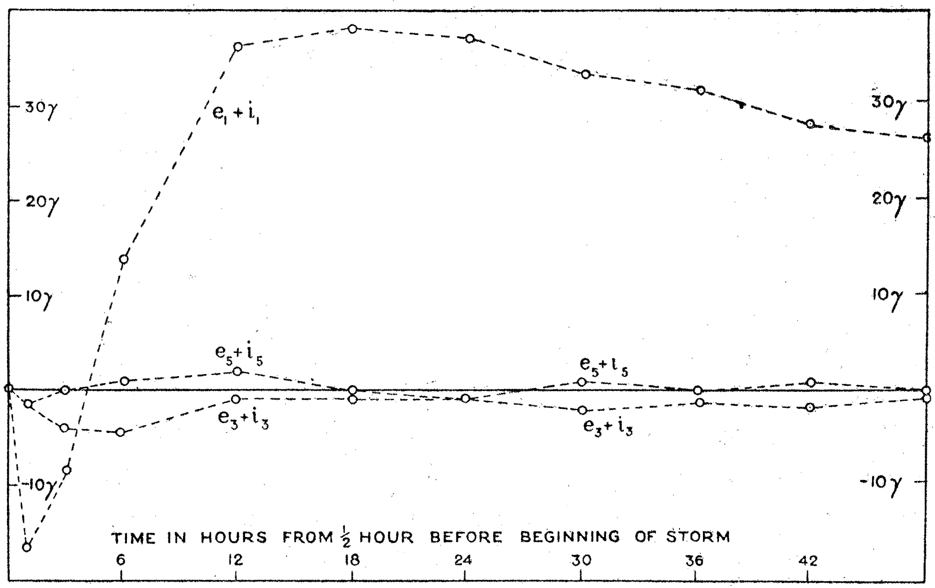

Fig. 1.

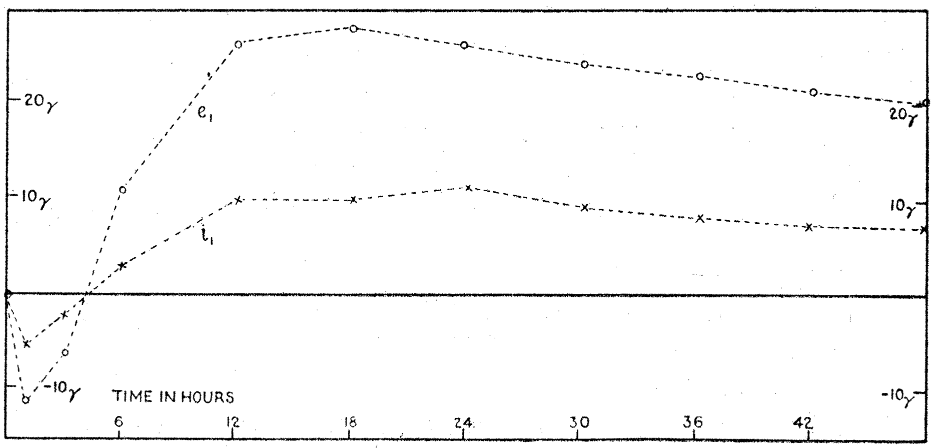

Fig. 2.

probably indicate the order of magnitude of the remainder of the storm-time field. At the equator ($\theta = 90°$),

$$(8) \qquad X = -(e_1 + i_1) + \tfrac{3}{2}(e_3 + i_3) - \tfrac{15}{8}(e_5 + i_5)\,; \text{ (equator)}\,;$$

hence, and from fig. 1, it appears that, from 0 h. to 4 h. (during which time $e_1 + i_1$ and $e_3 + i_3$ are both negative), P_3 opposes P_1, and reduces the initial increase of X; but from about 5 h. to 10 h., owing to the reversal of $e_1 + i_1$, P_3 reinforces P_1.

2.7. Table I and fig. 2 show that in the initial rapid change of X and Z, during the first half-hour of the storm, the internal part of P_1 is almost exactly half the external part: and that in the subsequent slower changes (from 1 h. to 12 h.) the ratio is rather less, that is, about 0·42.

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3. Electromagnetic and Magnetic Induction within a Uniform Conducting Permeable Sphere: General Formulæ.

3.1. In problems of electromagnetic and magnetic induction within a uniform conducting sphere, each spherical harmonic term V_n^p can be considered independently of all the others. Likewise, each periodic component of a particular harmonic, and each aperiodic component involving a real time factor e^{-lt}, can be separately treated.

3.2. Suppose that the earth has a uniform core, of radius $q\,a$, electric conductivity κ, and magnetic permeability μ. The part of the earth above this core is treated as non-conducting, and of permeability unity, though, in fact, the conductivity of the oceans is higher than that found for the core (§ 1).

3.3. *The daily magnetic variations.*—For the periodic component of frequency p, of the harmonic V_n^p, the complex ratio I_n^p/E_n^p, at the earth's surface $r = a$, on the assumption that the internal part of the field is due to electromagnetic and magnetic induction within the earth by the outer field, is given by*

$$(9) \qquad \frac{I_s}{E_s} = \frac{nq^{2n+1}}{n+1}\left[1 - \mu\left\{\frac{R_{n-1}}{R_n} + \frac{n(\mu-1)}{2n+1}\right\}^{-1}\right],$$

where I_s and E_s denote the surface values of I_n^p and E_n^p, while R_{n-1}/R_n may be derived from the formula

$$(10) \qquad \frac{R_{n-1}}{R_n} = \frac{\beta}{2n+1}\left[\left\{1 + \frac{n}{\beta} + \frac{n(n+1)}{4\beta^2} + \frac{0}{4\beta^3} + \ldots\right\} + i\left\{1 - \frac{n(n+1)}{4\beta^2} - \frac{n(n+1)}{4\beta^3} - \ldots\right\}\right],$$

when β, defined by

$$(11) \qquad \beta^2 = \frac{4\pi^2 p\,\kappa\,\mu q^2 a^2}{24\,.\,60\,.\,60},$$

is large; (10) is, in fact, an asymptotic expansion.

The values of I_s/E_s derived from observation (§ 2.4) indicate[3] that, if $\mu = 1$, β must be about $8\sqrt{p}$, while[4] if $\mu > 1$, κ and β must be increased nearly in the ratio $\mu : 1$. Computations were made for the two harmonics P_2^1 and P_3^2, and values of μ from 1 to 100 were considered.[4] It was concluded that κ and μ cannot be separately inferred from the periodic magnetic variations, but only their ratio κ/μ. This, however, did not completely dispose of the possibility that some particular value of μ might prove somewhat more suitable than other values, if all the harmonics discussed in (3) were considered, with more accurate determinations of I_s and E_s. The conclusion can be confirmed, however, without extensive computations, as follows: by (9) and (10),

$$(12) \qquad \frac{I_s}{E_s} = \frac{nq^{2n+1}}{n+1}\left\{\frac{(A-\mu) + iB}{A + iB}\right\},$$

* *Cf.* equations (5.7), (7.8), (7.10) and (7.12) of the memoir by Chapman and Whitehead, 'Trans. Camb. Phil. Soc.,' vol. 22 (1922). A slight correction is needed in (7.12), which corresponds to (10) above.

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where

(13) $$A = \frac{n(\mu - 1)}{2n+1} + \frac{\beta}{2n+1}\left\{1 + \frac{n}{\beta} + \frac{n(n+1)}{4\beta^2} + \frac{0}{4\beta^3} + \ldots\right\}$$
$$= \frac{n\mu + \beta}{2n+1} + \frac{n(n+1)}{2n+1}\frac{1}{4\beta} + \ldots$$

(14) $$B = \frac{\beta}{2n+1}\left\{1 - \frac{n(n+1)}{4\beta^2} - \frac{n(n+1)}{4\beta^3}\ldots\right\}.$$

Hence the amplitude ratio R and the phase difference ϕ between the internal and external parts of the field are given by

(15) $$R = \left|\frac{I_s}{E_s}\right| = \frac{nq^{2n+1}}{n+1}\left\{\frac{(A-\mu)^2 + B^2}{A^2 + B^2}\right\}^{\frac{1}{2}},$$

(16) $$\phi = \tan^{-1}\{B/(A-\mu)\} - \tan^{-1}(B/A).$$

Since it has been found that β is about $8\sqrt{p}$ when $\mu = 1$, and that κ, and consequently β (which is proportional to $\sqrt{(\mu\kappa)}$) must be increased if $\mu > 1$, (13) and (14) reduce approximately to

(17) $$A = \frac{n\mu + \beta}{2n+1}, \qquad B = \frac{\beta}{2n+1}$$

even when $\mu = 1$ and $n = 5$, while when $\mu > 1$ and $n < 5$ the approximation is closer. Substituting from (17) into (15), (16), we have

(18) $$R = \frac{nq^{2n+1}}{n+1}\left\{1 - \frac{(2n+1)(\beta/\mu - \frac{1}{2})}{1 + n\beta/\mu + \frac{1}{2}n^2}\right\}^{\frac{1}{2}},$$

(19) $$\tan\phi = \frac{(2n+1)\beta/\mu}{2\beta^2/\mu^2 - \beta/\mu - n(n+1)};$$

these involve κ and μ only in the form β/μ, which is proportional to $(\kappa/\mu)^{\frac{1}{2}}$.

The degree of accuracy of (18) and (19) can be illustrated by calculating ϕ from (19) and from (16) in the case[3] when $\mu = 1$, $\kappa = 3 \cdot 65.10^{-13}$, for the principal harmonic terms in the daily magnetic variation: the results are as follows :—

	P_2^1	P_3^2	P_4^3	P_5^4	P_1^1	P_2^2
	°	°	°	°	°	°
ϕ from (16) ..	18·9	18·7	19·3	20·5	11·6	13·5
ϕ from (19) ..	19·3	18·8	19·5	20·8	11·5	13·4

Since (19) is least accurate when $\mu = 1$, it follows that for $\mu > 1$ the formulæ (16) and (19) are practically equivalent, and that therefore it is not possible, from the daily magnetic variation, to do more than infer the *ratio* κ/μ, and not κ or μ separately.

3.4. *The storm-time variations.*—In this case the external inducing field is aperiodic,

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and initially and finally zero: its potential can be expressed as the sum of zonal harmonic terms $V_n = E_n (r^n/a^{n-1}) P_n (\cos\theta)$, where E_n is a function of the time. The induced field can be calculated by expressing E_n as the sum of a series of exponentials with respect to the time, or rather in the form

$$E_n = \sum_m E_{m,n} \{1 - \exp(-\lambda_{m,n} t)\}, \tag{20}$$

in which each term is initially zero; the constants $E_{m,n}$, $\lambda_{m,n}$ must be chosen (and a sufficient number of terms must be taken) to fit the function E_n with the desired accuracy.

It has been shown[9] that the induced field corresponding to the $E_{m,n}$ term in the harmonic V_n is given by

$$I_{m,n} (a^{n+2}/r^{n+1}) P_n (\cos\theta), \tag{21}$$

where

$$I_{m,n}/E_{m,n} \equiv \phi_{m,n}(t) = \frac{nq^{2n+1}}{n+1}\left[\sum_{s=1}^{\infty} \frac{2(2n+1)\mu\beta^2_{m,n}\{\exp(-\lambda_{m,n}t) - \exp(-l_{n,s}t)\}}{\{x^2_{n,s} + n(\mu-1)(n\mu+n+1)\}(x^2_{n,s} - \beta^2_{m,n})} - \frac{(n+1)(\mu-1)}{n\mu+n+1}\{1 - \exp(-\lambda_{m,n}t)\}\right]; \tag{22}$$

$x_{n,s}$ is the sth root* of the equation

$$x^{-n-\frac{1}{2}}\{xJ'_{n+\frac{1}{2}}(x) + (n\mu + \tfrac{1}{2}) J_{n+\frac{1}{2}}(x)\} = 0, \tag{23}$$

$J_{n+\frac{1}{2}}(x)$ being the Bessel function of the first kind and of order $n + \frac{1}{2}$; $l_{n,s}$ and $\beta_{m,n}$ are defined by

$$l_{n,s} = x^2_{n,s}/4\pi\kappa\mu q^2 a^2, \tag{24}$$

$$\beta^2_{m,n} = 4\pi\kappa\mu\lambda_{m,n} q^2 a^2. \tag{25}$$

The function $\phi_{m,n}(t)$ can be shown[9] to have the following properties† :—

(a) $\phi'(0) = nq^{2n+1}\lambda/(n+1)$, which is essentially positive: hence $I_{m,n}$ has initially the same sign as $E_{m,n}$.

(b) $\phi(t) \to -nq^{2n+1}(\mu - 1)/(n\mu + n + 1)$ as $t \to \infty$.

(c) $\phi(t)$ has only one maximum value, which lies on the graph of $\Phi(t)$, defined by

$$\Phi(t) \equiv \frac{nq^{2n+1}}{n+1}\left\{2(2n+1)\mu \sum_{s=0}^{\infty} \frac{\exp(-l_{n,s}t)}{x^2_{n,s} + n(\mu-1)(n\mu+n+1)} - \frac{(n+1)(\mu-1)}{n\mu+n+1}\right\}. \tag{26}$$

The function $\Phi(t)$ is the limit to which the function $\phi(t)$ tends as β (or λ) $\to \infty$,

* The roots are all real, and it is assumed that they are arranged in ascending order of magnitude.

† The suffixes m, n, are omitted in what follows, where no confusion is likely to result from the omission.

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i.e., when the change in the external field becomes instantaneous; $\Phi(t)$ decreases continuously from $nq^{2n+1}/(n+1)$ to $-nq^{2n+1}(\mu-1)/(n\mu+n+1)$ as t increases from zero to infinity. It follows that the graphs of $\Phi(t)$ and $\phi(t)$ are as shown in fig. 3. It will be observed that the maximum value of $\phi(t)$ will be smaller, the slower the rate of change of the external field.

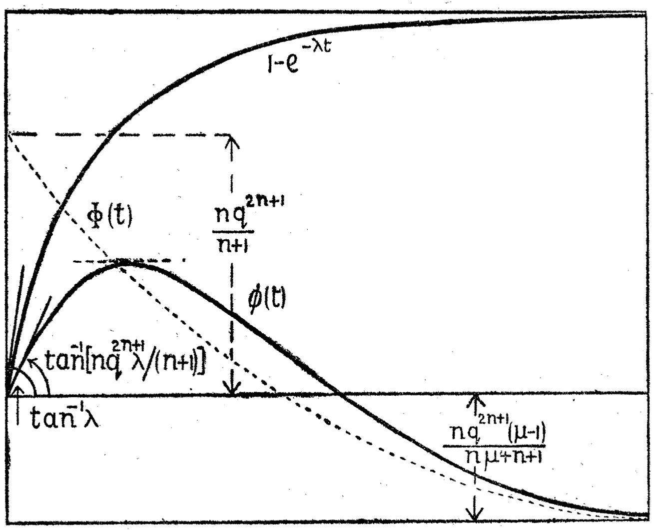

Fig. 3.

3.5. In the special case of unit permeability ($\mu=1$), the equations (22), (23), (26) reduce to

$$(27) \quad I_{m,n}/E_{m,n} = \phi_{m,n}(t) = \frac{2n(2n+1)q^{2n+1}}{n+1}\beta^2_{m,n}\sum_{s=1}^{\infty}\frac{\exp(-\lambda_{m,n}t)-\exp(-l_{n,s}t)}{x^2_{n,s}(x^2_{n,s}-\beta^2_{m,n})},$$

$$(28) \quad x^{-n-\frac{1}{2}}J_{n-\frac{1}{2}}(x) = 0,$$

$$(29) \quad \Phi_{m,n}(t) = 2(2n+1)\sum_{s=1}^{\infty}x^{-2}_{n,s}\exp(-l_{n,s}t).$$

Thus the roots $x_{n,s}$ are, in this case, the real zeros of the Bessel function $J_{n-\frac{1}{2}}$; also now $\phi(t)\to 0$ as $t\to\infty$.

3.6. Some general conclusions regarding the nature of the induced field, whatever the values of κ and μ, can be formed without detailed computation.

Every term in the infinite series in (22) is positive, for all values of t, because $x^2_{n,s}/\beta^2_{m,n} = l_{n,s}/\lambda_{m,n}$, by (24), (25), and therefore the numerator and denominator of each term have the same sign. Moreover, every term tends to zero as $t\to\infty$. Hence the corresponding part of $I_{m,n}$ has the same sign as $E_{m,n}$; as $t\to\infty$ it tends to zero, implying the eventual decay of the corresponding currents.

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The remaining term in (22) is not due to induced currents, but to induced magnetism in the permeable core; it vanishes if $\mu = 1$. This term is always negative, and bears a constant ratio to the time factor, $1 - \exp(-\lambda_{m,n}t)$, of the inducing field; it does not tend to zero as $t \to \infty$. Hence (if $\mu \neq 1$) it is ultimately more important than the field of the induced currents.

The general form of the function $\phi(t)$ has been indicated in fig. 3. At first it is positive, showing that the field of the induced currents outweighs that of the induced magnetism; but as the currents decay $\phi(t)$ ceases to increase; it attains a maximum which is always less than $nq^{2n+1}/(n+1)$; then it decreases, changes sign, and tends to the constant negative value $-nq^{2n+1}(\mu - 1)/(n\mu + n + 1)$, which is always (numerically) less than q^{2n+1}.

Hence, using (7), it follows that at first the internal field increases H and decreases V, at the surface, but that ultimately H is decreased and V increased, if $\mu > 1$.

If μ is fairly large (and q nearly equal to 1) the final influence of the induced magnetism is considerable, because $I_{m,n}/E_{m,n}$ tends approximately to $-q^{2n+1}$ as $t \to \infty$; the internal field then nearly cancels H, while its contribution to V is similar in magnitude to the external one (the ratio is approximately $(n+1)\,q^{2n+1}/n$, which may exceed unity).

During the early stages, however, the total induced magnetic field will be almost uninfluenced by the induced magnetism if the core is sufficiently conducting; for when $\kappa\mu q^2 a^2 \lambda$ is sufficiently large $\phi(t)$ is at first approximately equal to $nq^{2n+1}(1 - e^{-\lambda t})/(n+1)$, independent of μ. The field of the induced magnetism is just compensated by an increase of the induced currents, beyond what would be produced if $\mu = 1$.

Thus to determine μ it is necessary to consider the internal field in the later stages, when the induced currents have nearly died away.

3.7. The discussion in 3.6 refers only to the induced field corresponding to a particular harmonic term and time factor, as in (20). The same general conclusion will apply, however, with little or no change, to the field induced by any simple aperiodic primary field depending on a single harmonic. When the time factor (ψ) of the inducing field has several stationary values, that (ϕ) of the induced field will have one more of these, and the difference between any two consecutive stationary values of ϕ, at times t_1, t_2, will equal the algebraic sum of (i) a quantity less than $nq^{2n+1}/(n+1)$ times the difference between the corresponding stationary values of ψ, and (ii) a quantity depending on the decay, during the interval t_1 to t_2, of the field of the induced currents which existed at the time t_1. The second quantity will be appreciable only when the change in the external field is slow, and in this case the first quantity will be small.

3.8. To sum up briefly, we may say that the induced field will contain only those harmonics which are present in the inducing field; that, while the external field is varying, the coefficient (a function of t) of the "internal" harmonic of order n will be approximately $nq^{2n+1}/(n+1)$ times that of the corresponding "external" harmonic

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if κ is sufficiently large, and will be rather less than this if κ is smaller ; and finally that, if the external field remains comparatively steady over an interval sufficiently long for the induced currents to die away, then the final value of the coefficient of the internal harmonic of order n will be $-nq^{2n+1}(\mu-1)/(n\mu+n+1)$ times that of the corresponding external harmonic.

4. The Magnetic Permeability of the Core.

4.1. In 3.3 it has been shown that the induced part of the field of the daily magnetic variations is unlikely to afford information as to the separate values of κ and μ, but depends practically only on their ratio κ/μ.

In 3.6, 3.7, it has been shown that the ratio of the coefficients of the nth harmonic in the internal (induced) and external (primary) fields approximates to $nq^{2n+1}/(n+1)$, while the field is actively varying (*i.e.*, provided that $\kappa\mu q^2 a^2 \lambda$ is sufficiently large) ; this ratio, therefore, is independent both of κ and μ. Fig. 2 indicates that these conditions are realized during the first 48 hours of a magnetic storm, since the coefficient of the first internal harmonic ($n = 1$) is almost half that of the external harmonic. Hence it appears that μ cannot be determined from observations of the storm-time field during its active phase.

But since the storm-time field decays rather slowly, over several days, it may in its later stages indicate whether $\mu > 1$. As stated in 3.6, if μ exceeds unity, the H.F. (horizontal force) deviation should decrease more rapidly than the V.F. (vertical force) deviation ; for example, if $\mu = 10$, the ratio I_1/E_1, when the induced *currents* have died away, is $-q^3(\mu-1)/(\mu+2)$, or nearly $-\frac{3}{4}$ if q is nearly unity ; the H.F. deviation would be decreased to $\frac{1}{4}$ the value due to the external field alone, while the V.F. deviation would be increased in the ratio $\frac{5}{2}$.

A search for such an effect has been made, by examining the daily means of H.F. and V.F. at several observatories for nine or ten days following magnetic storms, but without success. The task is not easy, because the undisturbed state of the earth's field is not definite. It may be worth while to make a more detailed and extensive search for an induced magnetic field during the days following magnetic storms.

On physical grounds, however, it seems unlikely that μ appreciably exceeds unity. A larger value would presumably imply the existence of ferromagnetic material in the core ; but at ordinary pressures iron loses its permeability at 800° C., a temperature attained at about 100 km. depth[10]—less than the thickness of the relatively non-conducting outer shell. Increased pressure appears only to lower the temperature at which iron loses its permeability.[17]

Owing to our ignorance of the conditions deep within the earth, it is not inconceivable that the core is diamagnetic ($\mu < 1$). If so, the induced magnetism (though probably

[17] Adams and Green, 'Report Dep. Terr. Mag., Carnegie Inst.,' 1925–26, p. 225.

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very small) would reinforce the field of the induced currents; it might be possible to detect it in the last stages of the storm-time field—and also in the field of the daily magnetic variations, since the approximation (19) of 3.3 becomes increasingly inaccurate when μ decreases below unity; both possibilities are, however, rather remote.

In the remainder of this paper we shall assume that $\mu = 1$ throughout the conducting core.

5. Electromagnetic Induction by the Storm-Time Field; Numerical Values.

5.1. The following is the first attempt at an accurate calculation of the induced field due to the external part of the storm-time field, using probable values of κ and q. The first harmonic, which is of outstanding importance, is alone considered; the coefficient e_1 of its external part (*cf.* (6)) is indicated, as a function of time, in fig. 2.

5.2. It is necessary at the outset to represent e_1 by a suitable mathematical expression of the form $\Sigma\, a_m \{1 - \exp(-\lambda_m t)\}$; some latitude is possible in the choice of the numbers a_m and λ_m, but on physical grounds it is clear that if two formulæ fit the observed curve for e_1 equally well over a certain interval (from $t = 0$), the corresponding induced fields will be closely similar during the interval. The calculation of the induced field is much simplified if the λ's can be chosen from among the series of numbers l_1, determined by (23, 24) or (since we assume $\mu = 1$) (24, 28); (28) reduces to

(30) $$\sin x = 0$$

when $n = 1$, so that

(31) $$x_s = s\pi \quad (s = 1, 2, \ldots)$$

and

(32) $$l_s = s^2\pi^2/4\pi\,\kappa\, q^2 a^2 = s^2 \mathrm{A},$$

say, where

(33) $$\mathrm{A} = \pi/4\,\kappa\, q^2 a^2.$$

The values of κ and q found by Chapman are adopted in the first instance, *i.e.*,

(34) $$\kappa = 3{\cdot}65\,.\,10^{-13}\ \text{e.m.n.},\ q = 0{\cdot}96,$$

so that, since $a = 6{\cdot}37\,.\,10^8$ cm.,

(35) $$\mathrm{A} = 5{\cdot}74\,.\,10^{-6};$$

if we write $t = 8{\cdot}64\,.\,10^4\ \mathrm{T}$, so that T denotes time reckoned in the day as unit, we must replace A t by A′T, where A′ $= 0{\cdot}497$.

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It is found that e_1 can be represented by four terms, taking $\lambda_m = l_s$ for the values $s = 1, 3, 6, 8$, and taking $a_1 = 40 = -a_3$, $a_8 = 96 = -a_6$, so that, on reduction,

$$e_1 = -96\,(e^{-36\mathrm{A}t} - e^{-64\mathrm{A}t}) + 40\,(e^{-\mathrm{A}t} - e^{-9\mathrm{A}t}). \tag{36}$$

5.3. Price (*loc. cit.*) has shown that a primary external field of which the potential is

$$\{1 - \exp(-m^2 \mathrm{A}t)\}\, r\mathrm{P}_1(\cos\theta) \tag{37}$$

gives rise to an induced field of which the potential is

$$\phi_m \,.\, (a^3/r^2)\, \mathrm{P}_1(\cos\theta), \tag{38}$$

where

$$\phi_m = \frac{3q^3}{\pi^2}\left[\left\{\mathrm{A}t + \frac{21 - 2m^2\pi^2}{12m^2}\right\}\exp(-m^2\mathrm{A}t) - m^2 \sum_{s=1}^{\infty}{}' \frac{\exp(-s^2\mathrm{A}t)}{s^2(s^2 - m^2)}\right], \tag{39}$$

the term $s = m$ being omitted from the summation.

Thus for the potential $i_1\,(a^3/r^2)\,\mathrm{P}_1(\cos\theta)$ of the induced field corresponding to (36), we have

$$i_1 = -96\,(\phi_8 - \phi_6) + 40\,(\phi_3 - \phi_1). \tag{40}$$

5.4. The values of ϕ_m for $m = 1, 3, 6, 8$, computed from (35, 39), for times up to 96 hours, are given in Table II. The corresponding values of e_1 and i_1, calculated from (36) and (40), are given in Table III, and illustrated in fig. 4. The values of e_1 and i_1 determined from the observations (up to 48h.) are shown by dotted lines; the agreement between the observed and calculated values of e_1 is moderately good, showing that the chosen formula (36) is fairly satisfactory. The changes represented by the formula

Table II.—Values of ϕ_m, for $m = 1, 3, 6$ and 8.

Time in hrs.	1	2	3	6	9	12	15	18	21	24
ϕ_1	0·0303	0·0574	0·0822	0·1467	0·1995	0·2434	0·2795	0·3099	0·3346	0·3546
ϕ_3	0·2515	0·4377	0·5788	0·8213	0·9066	0·9142	0·8839	0·8383	0·7864	0·7349
ϕ_6	0·7718	1·0717	1·1733	1·1505	1·0218	0·9246	—	—	—	—
ϕ_8	1·0720	1·2684	1·2628	1·1165	0·9986	0·9066	—	—	—	—

Time in hrs.	30	36	42	48	64	96
ϕ_1	0·3832	0·3994	0·4059	0·4053	0·3797	0·2881
ϕ_2	0·6388	0·5558	0·4850	0·4244	0·3012	0·1543

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TABLE III.—The calculated values of the induced field for magnetic storm-time variations, on the assumption that $\kappa = 3\cdot65 \times 10^{-13}$ e.m.u., $q = 0\cdot96$ and $\mu = 1$. Coefficient of first harmonic in primary field denoted by e_1, and that for induced field by i_1.

Time.	1 hour.	2 hours.	3 hours.	6 hours.	9 hours.	12 hours.	15 hours.	18 hours.
	γ							
e_1	− 14·0	− 4·1	+ 6·3	21·2	25·6	26·9	26·8	26·1
i_1	− 5·4	− 1·0	+ 3·0	6·4	7·0	6·8	6·1	5·9

Time.	21 hours.	24 hours.	30 hours.	36 hours.	42 hours.	48 hours.	64 hours.	96 hours.
	γ							
e_1	25·1	24·0	21·3	18·9	16·8	14·8	10·6	5·5
i_1	4·9	4·1	2·8	1·7	0·9	0·2	−0·8	−1·4

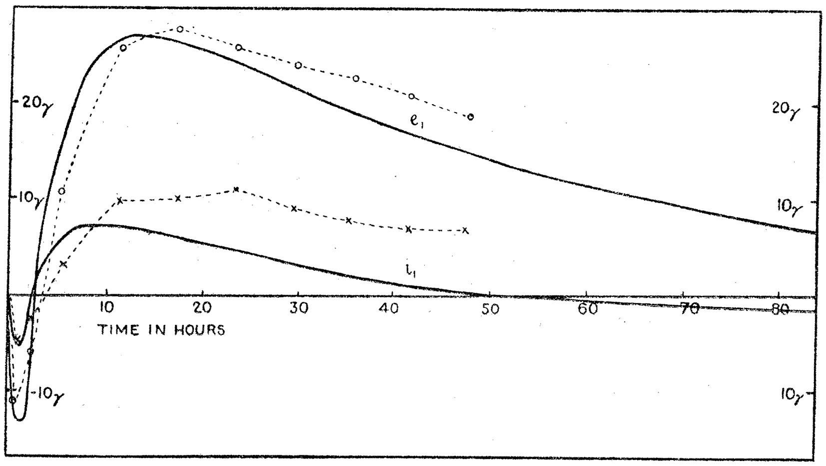

Fig. 4.

are slightly too rapid during the first six or eight hours, and the maximum is attained somewhat too soon; the subsequent slow decrease is likewise too rapid, so that e_1 as calculated is somewhat too small (it may be noted that CHAPMAN and WHITEHEAD, in representing the storm-time variations, took $e^{-0\cdot805T}$ as the most slowly varying term, instead of $e^{-0\cdot507T}$ as here).

The calculated curve for i_1 differs from the observed curve in a similar sense; but the discrepancy during the second day of the storm is far greater than for e_1. Moreover, if the formula (36) were modified to fit e_1 better, the discrepancy in i_1 would not be reduced but increased, because a more slowly varying outer field induces a less intense inner field.

This indicates that the values (34) for κ and q, which accord well with the relation between the induced and inducing fields in the case of the diurnally periodic variations,

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are not consistent with it in the case of the slower, aperiodic, variation in the later stages of a magnetic storm. While it was natural and proper, in the first instance, when considering the daily magnetic variations alone, to try to represent the earth by a uniformly conducting sphere, surrounded by an outer non-conducting layer, it now appears that this simple model is inconsistent with the observed facts, when both the daily and the storm-time variations are considered.

5.5. The discrepancy with the simple model (34) is further illustrated by fig. 5, which refers to a combined field (primary field plus calculated induced field) chosen so as to fit the observed H.F. variations (dotted lines) fairly well; the corresponding V.F. variations are much greater than is observed; this indicates that the calculated ratio of the induced to the primary field is too small, agreeing with fig. 4.

5.6. The induced field would be increased, as the observations require, if a larger value were assigned to κ; calculations have shown that, with $q \leq 0 \cdot 96$, κ must be

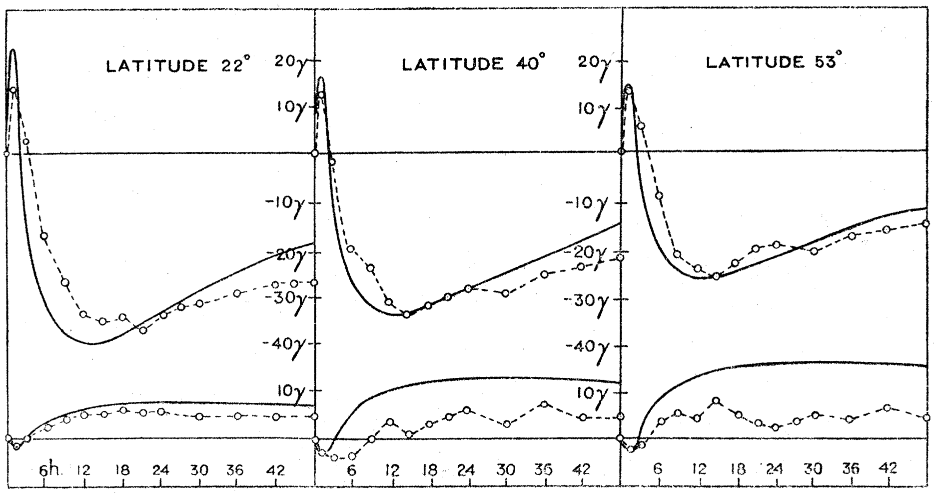

Fig. 5.

increased at least to 44.10^{-13} (*i.e.*, 12-fold) to fit the storm-time observations; the smaller the value of q, the larger must be the value of κ, for two reasons, namely, because of the factor q^3 arising from the variation of the inducing and induced fields across the non-conducting outer layer, and further, because in the expression for the induced currents κ is associated with a factor q^2. Table IIIA shows the values of the induced field, calculated on the assumption that $\kappa = 44.10^{-13}$, $q = 0 \cdot 94$, $\mu = 1$; the value $q = 0 \cdot 94$ has been taken because the diurnal variations clearly indicate that at $q = 0 \cdot 96$ the value of κ is only about 4.10^{-13}. The initial phase has been ignored in these calculations. The first line shows the values of e_1, referring to the external field, calculated from the expression chosen to represent the observed values of e_1; it will be seen to agree fairly well, after twelve hours, with the actual values, shown in line 2. Lines 3 and 4 show the calculated and actual values of the coefficient i_1 of the induced field. The value $\kappa = 44.10^{-13}$ with $q = 0 \cdot 94$ appears to be about right; a decrease of 10 per

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cent. would probably make the induced field too small. This value is considerably larger than the value ($5 \cdot 6.10^{-13}$) previously estimated by CHAPMAN and WHITEHEAD (*loc. cit.*) when considering in a rougher way the slowest exponential term in e_1. The considerable difference between the value $\kappa = 44.10^{-13}$ and that derived by CHAPMAN from the diurnal variations ($3 \cdot 65.10^{-13}$) appears to indicate a rapid increase in κ as we descend in the earth.

TABLE IIIA.—The calculated values of the induced field for magnetic "storm-time" variations, when $\kappa = 44.10^{-13}$ e.m.u., $q = 0 \cdot 94$, $\mu = 1$, compared with the values obtained from the observations.

Time in hours	0	6	12	18	24	30	36	42	48
45 {exp (− 9A′T) − exp (− 64 A′T)}	0	18·2	25·4	27·9	27·8	26·2	25·8	23·4	21·1
e_1 from observations	0	11	26	28	26	24	23	21	20
i_1 calculated	0	6·9	9·3	10·4	9·4	8·5	7·8	6·7	6·2
i_1 from observations	0	3	10	10	11	9	8	7	7

The further development of the theory will require consideration of a more complicated core, in which κ varies continuously, or, if found desirable for mathematical convenience, discontinuously from one layer to another. Before extending the calculations in this way, as it is our intention to do in a further paper, it is desirable to gain some idea of the probable rate of variation of κ with depth, in the region where the induced currents flow. One step towards this is to examine the depth-distribution of the induced currents in the two cases of the daily and the storm-time variations, using the original simple model of the earth, given by (34). This will also serve to indicate the depths for which these magnetic variations are likely to afford information.

6. THE DISTRIBUTION OF INDUCED CURRENTS WITHIN THE EARTH.

(*a*) *The Diurnal Variations.*

6.1. The current-density $\mathbf{i}$ at an internal point P, given by r, θ, λ, due to induction by the external field represented by the real part of

(41) $$E_n^{p,\alpha} r^n a^{-n+1} e^{\iota\alpha t} P_n^p (\cos\theta) \cos(p\lambda + \sigma),$$

$E_n^{p,\alpha}$ being a complex number, is given[9] by the real part of

(42) $$\mathbf{i} = \frac{-\iota\kappa\alpha E_n^{p,\alpha} e^{\iota\alpha t} R_n (k^2 r^2)}{(n+1)\, a^{n-1} R_{n-1} (k^2 q^2 a^2)} \mathbf{r} \wedge \mathbf{grad} \{r^n P_n^p (\cos\theta) \cos(p\lambda + \sigma)\};$$

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where **r** denotes the position vector of the point relative to the earth's centre, and $\wedge$ is the sign used for vector-multiplication of vectors, while

(43) $$k^2 = -4\pi\kappa\iota\alpha,$$

(44) $$\mathrm{R}_n(z^2) = 2^{n+\frac{1}{2}}\Gamma(n+\tfrac{3}{2})\, z^{-n-\frac{1}{2}}\mathrm{J}_{n+\frac{1}{2}}(z)$$
$$= 1 - \frac{z^2}{2(2n+3)} + \frac{z^4}{2.4.(2n+3)(2n+5)} - \ldots$$

6.2. The current-intensity at **r** is given by the modulus of (42), and its phase by the argument of (42) with the factor $e^{i\alpha t}$ removed. Since k is complex, and therefore also $\mathrm{R}_n(k^2r^2)$, the phase as well as the intensity of the currents varies with depth. To determine this variation it is necessary to separate the real and imaginary parts of $\mathrm{R}_n(k^2r^2)$.

It is convenient to write

(45) $$x = \iota^{\frac{1}{2}}kr = (4\pi\kappa\alpha)^{\frac{1}{2}}\, r = (4\pi\kappa q^2a^2\alpha)^{\frac{1}{2}}\,\rho,$$

where

(46) $$\rho = r/qa,$$

so that ρ denotes r or OP expressed as a fraction of the radius (qa) of the conducting core. When α is given by (5a), as in the cases here considered, and when the constants (34) are adopted for the core,

(47) $$x = 11{\cdot}17\,\rho p^{\frac{1}{2}}.$$

Let the real and imaginary parts of $\mathrm{R}_n(k^2r^2)$, or $\mathrm{R}_n(-ix^2)$, be denoted by $\mathrm{R}_n{}^r$ and $\mathrm{R}_n{}^i$ respectively. Then from (44) we obtain

(48) $$\mathrm{R}_0{}^r = (1/2x)\{\exp(x/\sqrt{2})\cos(x/\sqrt{2}-\pi/4) + \exp(-x/\sqrt{2})\sin(x/\sqrt{2}-\pi/4)\},$$

(49) $$\mathrm{R}_0{}^i = (1/2x)\{\exp(x/\sqrt{2})\sin(x/\sqrt{2}-\pi/4) + \exp(-x/\sqrt{2})\cos(x/\sqrt{2}-\pi/4)\},$$

(50) $$\mathrm{R}_1{}^r = (3/2x^2)\{\exp(x/\sqrt{2})\sin x/\sqrt{2} - \exp(-x/\sqrt{2})\sin x/\sqrt{2}\}$$
$$+ (3/2x^3)\{-\exp(x/\sqrt{2})\sin(x/\sqrt{2}-\pi/4) - \exp(-x/\sqrt{2})\cos(x/\sqrt{2}-\pi/4)\},$$

(51) $$\mathrm{R}_1{}^i = (3/2x^2)\{-\exp(x/\sqrt{2})\cos x/\sqrt{2} - \exp(-x/\sqrt{2})\cos x/\sqrt{2}\}$$
$$+ (3/2x^3)\{\exp(x/\sqrt{2})\cos(x/\sqrt{2}-\pi/4) + \exp(-x/\sqrt{2})\sin(x/\sqrt{2}-\pi/4)\},$$

while for greater values of n, the corresponding values of $\mathrm{R}_n{}^r$ and $\mathrm{R}_n{}^i$ are given by the recurrence formulæ

(52) $$\mathrm{R}_n{}^r = -\frac{4n^2-1}{x^2}[\mathrm{R}^i{}_{n-1} - \mathrm{R}^i{}_{n-2}],$$

(53) $$\mathrm{R}_n{}^i = \frac{4n^2-1}{x^2}[\mathrm{R}^r{}_{n-1} - \mathrm{R}^r{}_{n-2}].$$

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6.3. The values of the modulus and argument of $R_n(k^2r^2)$, for the values $p = 1, 2, 3, 4$, and $n = p, p+1$, at various depths ρ, are given in Table IV.

TABLE IV.—Moduli and Arguments of $R_n(-ix^2)$, where $x = 11 \cdot 17\, \rho\, p^{\frac{1}{2}}$.

	$\rho =$	1·0	0·9	0·8	0·7	0·6	0·5
$p = 1$	Mod. R_0 .	120·5					
	Arg. R_0 .	47·5°					
	Mod. R_1 .	30·4	16·7	9·64	5·65	3·44	2·20
	Arg. R_1 .	6·3°	— 38·5°	— 83°	— 127·5°	— 172°	— 215·5°
	Mod. R_2 .	12·0	7·31	4·60	3·02	2·15	1·52
	Arg. R_3 .	— 31°	— 75°	— 118·5°	— 161°	— 203·5°	— 244°
$p = 2$	Mod. R_1 .	408					
	Arg. R_1 .	— 175·5°					
	Mod. R_2 .	118·2	52·2	23·8	11·4	5·74	3·10
	Arg. R_2 .	—207°	— 270°	27°	— 35·5°	— 97°	— 158°
	Mod. R_3 .	45·8	22·1	11·2	5·94	3·37	2·08
	Arg. R_3 .	—244°	54°	— 8°	— 68·5°	— 128·5°	— 186·5°
$p = 3$	Mod. R_2 .	812					
	Arg. R_2 .	— 64·5°					
	Mod. R_3 .	263	99	39·7	16·3	7·40	3·72
	Arg. R_3 .	257°	179°	104°	27·5°	—47·4°	— 120°
	Mod. R_4 .	106	44	19·1	8·80	4·40	2·36
	Arg. R_4 .	221°	145°	70°	— 5·2°	— 77·5°	— 147°
$p = 4$	Mod. R_3 .	$1 \cdot 26 \times 10^3$					
	Arg. R_3 .	16·3°					
	Mod. R_4 .	$4 \cdot 48 \times 10^2$	$1 \cdot 51 \times 10^2$	53·6	20·3	8·38	3·85
	Arg. R_4 .	— 21·4°	— 109·7°	— 197·5°	— 284·5°	— 370·4°	— 454·3°
	Mod. R_5 .	$1 \cdot 87 \times 10^2$	69·0	27·2	11·4	5·2	2·8
	Arg. R_5 .	— 57°	— 144·3°	—231·0°	44·0°	— 39·4°	— 120·4°

By (42) the amplitude of the (periodic) current intensity at P is a certain constant multiple of mod $R_n(k^2r^2)\, \mathbf{r} \wedge \mathbf{grad}\, r^n P_n^p(\cos\theta)\cos(p\lambda + \sigma)$.

The magnitude of the vector product is proportional to r^n, and its direction is the same along any radial line OP. Hence the amplitude of the periodic variation of $\mathbf{i}$ at P is proportional to r^n mod $R_n(k^2r^2)$, and the ratio (f) of its amplitude to that at the point P_0 in which OP cuts the surface of the core is therefore

(54) $$\rho^n \bmod \{R_n(k^2r^2)/R_n(k^2q^2a^2)\} \equiv f.$$

The values of f determined from Table IV and (54) are given in Table V.

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TABLE V.

Depth below surface of earth in miles.	Depth below surface of core ($r = qa$)	P_2^1 (−18·9°)		P_3^2 (−18·7°)		P_4^3 (−19·3°)		P_5^4 (−20·5°)		P_1^1 (−11°)		P_2^2 (−13°)		P_3^3 (−15°)		P_4^4 (−17°)	
		f	ϵ	f	ϵ	f	ϵ	f	ϵ	f	ϵ	f	ϵ	f	ϵ	f	ϵ
160	0	1·0	− 53°	1·0	− 53°	1·0	− 54°	1·0	− 54°	1·00	− 49°	1·0	− 58°	1·0	− 51°	1·0	− 52°
540	$0 \cdot 1qa$	0·493	−9°	0·351	9°	0·273	22°	0·218	33°	0·494	− 4°	0·358	5°	0·274	26°	0·221	36°
930	$0 \cdot 2qa$	0·245	34°	0·125	71°	0·074	97°	0·048	120°	0·254	41°	0·129	68°	0·077	102°	0·049	124°
1310	$0 \cdot 3qa$	0·123	77°	0·045	132°	0·023	172°	0·012	205°	0·130	85°	0·048	130°	0·023	178°	0·012	211°
1690	$0 \cdot 4qa$	0·064	119°	0·016	192°	0·006	244°	0·002	288°	0·068	129°	0·018	192°	0·006	253°	0·002	297°
2080	$0 \cdot 5qa$	0·032	160°	0·006	250°	0·002	314°	0·000	369°	0·036	173°	0·007	253°	0·002	326°	0·001	382°

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6.4. Table V also contains values of ε, the difference of phase between the south component ($-X_e$) of the external field, and the easterly current intensity at the level ρ. The value of $-X_e$ is, by (7),

$$-E_n^{p,a} e^{\iota at} \frac{dp_n^p(\cos\theta)}{d\theta} \cos(p\lambda + \sigma).$$

Since the vector product in (42) is perpendicular to **r**, the current **i** is everywhere confined to concentric spherical shells. The easterly component of **i** is

$$(55) \qquad -\frac{\iota\kappa\alpha E_n^{p,a} e^{\iota at}}{(n+1)\, a^{n-1} R_{n-1}(k^2q^2a^2)} R_n(k^2r^2)\, kr^n \frac{dp_n^p(\cos\theta)}{d\theta} \cos(p\lambda + \sigma).$$

Hence, comparing the arguments of the two expressions, and omitting real factors of the same sign in the two formulæ, we have

$$(56) \qquad \begin{aligned} \varepsilon &= -\arg\{\iota R_n(k^2r^2)/R_{n-1}(k^2q^2a^2)\} \\ &= \arg R_{n-1}(k^2q^2a^2) - \arg R_n(k^2r^2) - 90^\circ \end{aligned}$$

6.5. In fig. 6, the full lines represent f as a function of depth, for the four harmonics P_2^1, P_2^2, P^3, P_5^4; Table V shows that f is almost the same for P_p^p and P^p_{p+1}. The

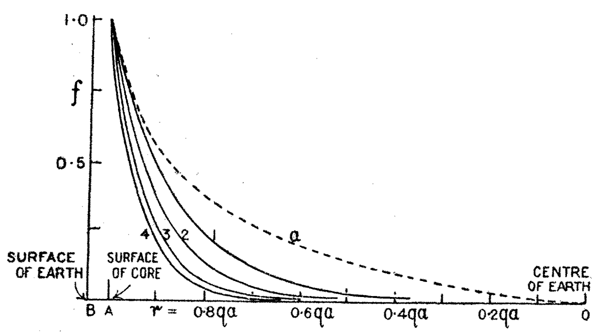

Fig. 6.

deeper penetration of the currents induced by the more slowly varying fields is well shown; thus the current-intensity falls to one-tenth its surface intensity at a depth $0 \cdot 3\ qa$ for the 24-hourly harmonics P_1^1 and P_2^1, while for the 6-hourly harmonics P_4^4 and P_5^4 the depth is only $0 \cdot 14\ qa$ below the surface of the core.

6.6. Table V shows that the value of ε at the surface of the core is about 50° for all the eight harmonics, and that ε increases with depth, at a rate which is nearly independent of n, but increases considerably with p; for P_1^1 and P_2^1 it is about 43° per $\frac{1}{10}\ qa$ increase of depth, while for P_4^4 and P_5^4 it is about 85°; the rate is nearly independent of depth (down to $\frac{1}{2}\ qa$), but decreases very slightly as we go downwards. The change of phase,

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as well as of intensity, of the induced currents with increasing depth is illustrated by fig. 7, which refers to the harmonic P_3^2.

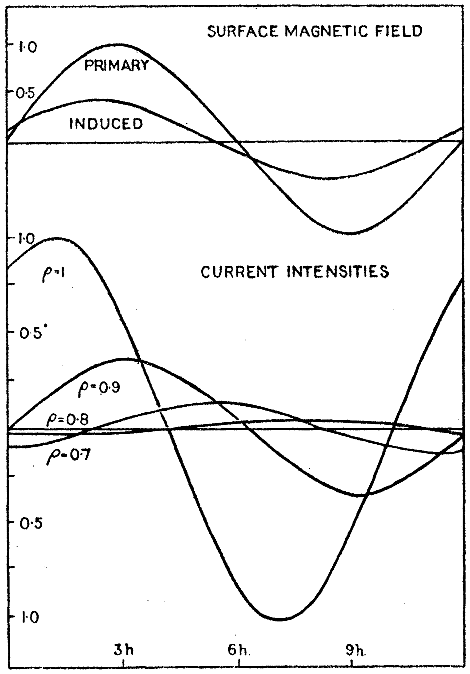

Fig. 7.

6.7. The field of the induced currents at the earth's surface ($r = a$) is made up of contributions from the currents at all depths within the core ; the contribution of the deeper layers is small not only because of the downward decrease of the current-intensity, but also because of their increasing distance from the surface. Consider a current-shell of infinitesimal thickness d and of radius ρqa ; it is readily seen, from considerations of dimensions* or otherwise, that the intensity of its own magnetic field at a point just outside its surface is independent of the radius, being determined solely by the value and distribution of $\mathbf{i}$ over the shell. The potential† of the magnetic field at external points will vary as $r^{-(n+1)}$, and the intensity will therefore vary as $r^{-(n+2)}$. Hence at the earth's surface the intensity of the field due to the shell will be proportional to $(\rho qa/a)^{n+2}\,\mathbf{i}$, and its ratio to the intensity due to a shell of equal thickness, situated at the surface of the core, is $(\rho q)^{n+2} f$. Values of this ratio are given in Table VI.

* The dimensions of the magnetic field intensity and of the current intensity (integrated over the thickness of the current sheet) are the same, while the only linear dimension that can enter into the relation between them is the radius of the sphere, whence the result follows. The result may also be proved directly from the boundary conditions.

† If the medium surrounding the current sheet is conducting, the *total* magnetic field at any point in the medium does not possess a scalar potential, but we are concerned with the field of the current sheet only, and this does possess a potential.

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TABLE VI.—The Relative Values of the maximum contributions to the surface magnetic field from currents in layers of equal (infinitesimal) thickness at various depths.

ρ	P_1^1	P_2^1	P_2^2	P_3^2	P_3^3	P_4^3	P_4^4	P_5^4	P_1 (*cf.* § 7.5) (aperiodic).
1·0	1	1	1	1	1	1	1	1	1
0·9	0·359	0·324	0·235	0·207	0·162	0·145	0·118	0·104	0·422
0·8	0·125	0·100	0·053	0·041	0·025	0·019	0·013	0·010	0·198
0·7	0·042	0·034	0·013	0·009	0·004	0·003	0·002	0·001	0·091
0·6	0·014	0·008	0·002	0·001					0·043
0·5	0·004	0·002							0·019

6.8. By integrating $(\rho q)^{n+2} f$ over successive layers of the core each of thickness $\frac{1}{10}$ qa, and expressing the result as a percentage of the integral over the whole range ($\rho = 1$ to $\rho = 0$), an approximate idea is obtained of the relative intensities of the fields of these layers at the earth's surface; the results are given in Table VII. It appears that about 90 per cent. of the surface field comes from the currents in the upper part of the core, down to a depth of $\frac{1}{5}$ qa or less. The currents at a depth of half the radius of the core, or below this level, are of no importance in determining the induced field at the surface.

TABLE VII.—The relative intensities of surface magnetic fields due to the currents in shells each of thickness $0\cdot 1qa$.

Shell.	P_1^1	P_2^1	P_2^2	P_3^2	P_3^3	P_4^3	P_4^4	P_5^4	P_1 (*cf.* § 7.5) (aperiodic).
1·0–0·9	66	70	77	78	82	84	86	89	56
0·9–0·8	23	22	18	17·5	16	14	13	11	25·5
0·8–0·7	8	6·5	4	4	2	2	1		11·5
0·7–0·6	2	1·5	1	0·5					5
0·6–0·5	1								2
Sum	100	100	100	100	100	100	100	100	100

6.9. The results in Table VII are not quite accurate, because in integrating through each layer, the continuous variation of phase with depth should be taken into account; the error is small, however, because the rate of change of ε, and the proportional rate of change of f, are nearly independent of depth. But in considering not merely the relative intensities of the surface fields due to the several layers, but their actual contributions to the whole induced field at the surface, the differences (δ) of phase between the resultant induced field, and the partial fields of the different layers, should be taken account of, by applying the factor cos δ appropriate to each layer.

The phases ε_0 of the resultant induced fields, relative to the external primary field, are given in the first row of Table VI (they are the arguments of I_s/E_s : *cf.* (9)). They

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represent a weighted mean of the phases ε of the current-sheets at the different depths. Since the difference of mean phase increases by about 90 per cent. on descending through $\frac{1}{5}qa$ for P_1^1 or P_2^1, and through $\frac{1}{10}qa$ for P_4^4 or P_5^4, it is clear that the contribution of the third layer from the top in the former case, and the second layer in the latter, when combined with the contribution of the first layer, will scarcely alter its intensity, but will merely increase the phase. Still lower layers will actually reduce the intensity, but by a very small amount because of the rapid downward decrease of $(\rho q)^{\rho+2}f$.

The relative actual contributions of the various layers to the *intensity* of the surface induced field are given in Table VIII. This is obtained by constructing a Table (analogous to VI) of $(\rho q)^{n+2}f\cos\delta$, and then integrating as in calculating Table VII; the layers are, however, taken of half the thickness, namely $\frac{1}{20}qa$. The same table shows the corresponding values of the "transverse" components of the contributions, expressed as fractions of the resultant intensity; it is obtained by integrating $(\rho q)^{n+2}f\sin\delta$ over the successive layers, and dividing by the complete integral of $(\rho q)^{n+2}f\cos\delta$; the positive sign indicates a contribution which tends to advance the phase of the resultant induced field.

7. The Distribution of the Induced Currents within the Core.

(*b*) *The Storm-time Variations.*

7.1. An external field having the potential function (*cf.* (20))

$$E_{m,n}\{1-\exp(-\lambda_{m,n}t)\}(r^n/a^{n-1})P_n(\cos\theta) \tag{57}$$

induces currents which flow solely in the easterly horizontal direction at each point; the current-density is [9]

$$\left\{-\frac{R_n(\beta^2_{m,n}r^2/q^2a^2)}{R_{n-1}(\beta^2_{m,n})}\exp(-\lambda_{m,n}t) + 2(2n+1)\sum_{s=1}^{\infty}\frac{R_n(x^2_{n,s}r^2/q^2a^2)\exp(-l_{n,s}t)}{(x^2_{n,s}-\beta^2_{m,n})R_n(x^2_{n,s})}\right\}\frac{\kappa\lambda_{m,n}E_{m,n}r^n}{(n+1)q^{n-1}a^{n-1}}\frac{dP_n(\cos\theta)}{d\theta} \tag{58}$$

in the notation of § 3 and § 6.

When $\lambda_{m,n}$ is equal to one of the numbers $l_{n,s}$, say $l_{n,m}$, so that also $\beta_{m,n}=x_{n,m}$, the term $s=m$ must be omitted from the summation, and the first term in the bracket must be replaced by

$$\left\{\frac{(2n+1)\{2R_{n-1}(z)-R_,(z)\}}{4x^2_{n,m}}-\frac{R_n(z)t}{4\pi\kappa\mu q^2a^2}\right\}\frac{2(2n+1)}{R_n(x^2_{n,m})}\exp(-l_{n,m}t), \tag{59}$$

where $z=x^2_{n,m}r^2/q^2a^2$.

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TABLE VIII.—The relative contributions of the various layers to the intensity of the surface induced field.

Layer.	P_1^1		P_2^1		P_2^2		P_3^2		P_3^3		P_4^3		P_4^4		P_5^4	
	Direct.	Transverse.	Direct.	Transverse.	Direct.	Transverse.	Direct.	Transverse.	Direct.	Transverse.	Direct.	Transverse.	Direct.	Transverse.	Direct.	Transverse.
$\rho =$ 1·0—0·95	0·443	+0·221	0·465	+0·200	0·542	+0·230	0·606	+0·235	0·690	+0·241	0·688	+0·255	0·748	+0·227	0·759	+0·218
0·95—0·90	0·299	+0·030	0·288	+0·020	0·317	—0·014	0·286	—0·029	0·262	—0·089	0·258	—0·100	0·240	—0·104	0·222	—0·115
0·90—0·85	0·172	—0·057	0·166	—0·064	0·133	—0·091	0·107	—0·100	0·070	—0·098	0·075	—0·103	0·040	—0·093	0·043	—0·084
0·85—0·80	0·083	—0·069	0·080	—0·065	0·036	—0·074	0·024	—0·070	0·000	—0·045	0·000	—0·045	0·008	—0·029	—0·006	—0·018
0·80—0·75	0·026	—0·061	0·024	—0·044	—0·004	—0·036	—0·003	—0·028	—0·014	—0·011	—0·013	—0·008	—0·015	—0·001	—0·014	
0·75—0·70	0·002	—0·036	0·001	—0·028	—0·011	—0·014	—0·009	—0·008	—0·008		—0·008		—0·005		—0·005	
0·70—0·65	—0·006	—0·020	—0·005	—0·015	—0·009	—0·001	—0·008									
0·65—0·60	—0·008	—0·008	—0·008	—0·003	—0·003		—0·003									
0·60—0·55	—0·007		—0·007													
0·55—0·50	—0·005		—0·005													

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Only the first harmonic will be considered, and since (§ 5.2)

$$x_{1,s} = s\pi,$$

$\lambda_{m,n} = Am^2$, and (58) takes the special form

(60) $$\Big[(-1)^m \{2R_0(m^2\pi^2\rho^2) - R_1(m^2\pi^2\rho^2) - \tfrac{4}{3}Am^2tR_1(m^2\pi^2\rho^2)\}\tfrac{3}{16}\pi m^2\rho \exp(-Am^2t) + \sum_{s=1}^{\infty}{}' \frac{\pi m^2\rho(-1)^s s^2}{4(s^2-m^2)} R_1(s^2\pi^2\rho^2)\exp(-As^2t)\Big]\frac{E_{m,1}\sin\theta}{qa} \equiv \frac{E_{m,1}\sin\theta}{qa} I(m,\rho).$$

The functions R_0 and R_1 are as follows :—

(61) $$R_0(s^2\pi^2\rho^2) = \frac{\sin(s\pi\rho)}{s\pi\rho},$$

(62) $$R_1(s^2\pi^2\rho^2) = \frac{3}{s^2\pi^2\rho^2}\left\{\frac{\sin(s\pi\rho)}{s\pi\rho} - \cos(s\pi\rho)\right\}.$$

7.2. The formula (60) has been used to calculate the current distribution due to the "external" first harmonic

(63) $$40(e^{-At} - e^{-9At})\, r\cos\theta \equiv e_1 r\cos\theta,$$

which corresponds to the second part of (36). The first part of (36) is not considered in this section of our work, in order to reduce the labour of computation; the first part represents the initial rapid phase of the storm-time changes, which induces a fleeting, shallow current system in the core, whereas we are here chiefly interested in the more enduring and deeper lying current system induced by the main phase of the storm-time field.

7.3. The graph of $40(e^{-9At} - e^{-At})$ is represented in the upper part of fig. 8; if t is measured from the fourth hour after the commencement of the storm, this graph (marked e_1) agrees fairly well with the observed values of the external first harmonic (indicated by small circles). During the second day (63) decreases rather too rapidly, but it agrees well with observation during the first day, which is the period of chief importance for our immediate purpose.

The coefficient i_1 of $(a^3/r^2)\cos\theta$ in the potential of the induced field was calculated as in § 5, and its graph is shown as i_1 in fig. 8.

7.4. The induced current-intensity in the core is

(64) $$\frac{40\sin\theta}{qa} I(\rho) \equiv \frac{40\sin\theta}{qa}\{I(1,\rho) - I(3,\rho)\}.$$

The values of $I(\rho)$ calculated from (60, 64) for various depths in the core and for times

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up to 96 hours are given in Table IX and illustrated in fig. 8; Table IX also contains the corresponding values of e_1 and i_1.

Fig. 8 shows that the induced currents near the *surface* of the core rapidly attain their maximum about ten hours before the inducing field reaches its maximum; by

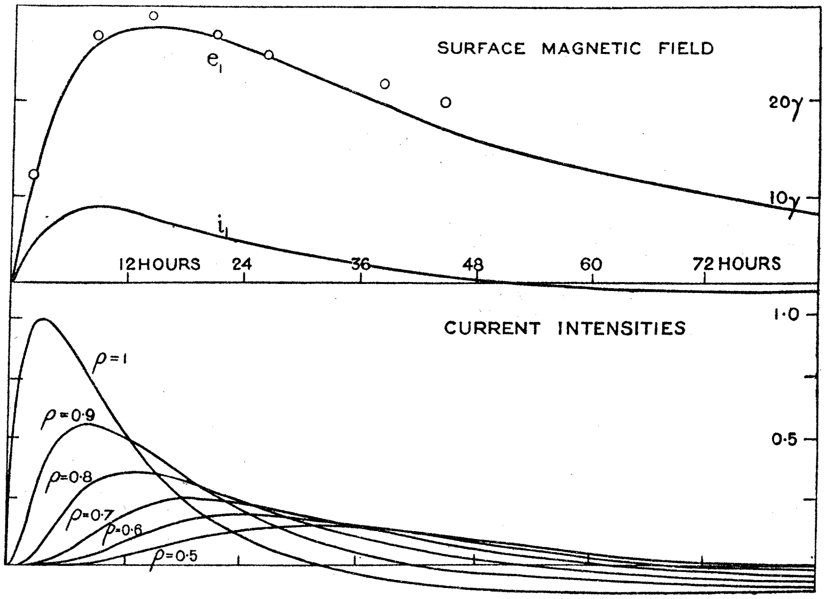

Fig. 8.

this time the surface currents have fallen to half their maximum value; they fall to zero at about 33h, and attain a small negative maximum at about 64h, afterwards tending slowly to zero. Similar current-changes occur at points within the core, but more slowly, and with smaller current intensities, as we go downwards.

Fig. 9 illustrates the changing distribution of current in another way, showing the current-intensity as a function of depth (from S the surface to O the centre of the core) at different times. The external field may be said to produce, by its rapid rise, a pulse of current which is transmitted inwards, with decreasing intensity; and later, by its slow decay, a much weaker pulse of the opposite sign.

7.5. The relative values of the (non-simultaneous) maximum contributions of successive current-layers to the induced field at the earth's surface have been calculated as in § 6.8. and are given in Table VII; the intermediately calculated quantities relating to thin shells are given in Table VI, *i.e.*, the values of $(\rho q)^3 f$, where f is the ratio of the maximum ordinate of the current-curve at the level ρ, in fig. 8, to the maximum ordinate for the surface current ($\rho = 1$).

It appears from Table VII that the currents induced by the storm-time field, at depths as great as one-third the radius of the core, would (if κ were uniform) make measurable contributions to the surface field.

7.6. The current system induced by the storm-time field differs from that of the daily magnetic variations, particularly in that the relative importance of the deeper

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TABLE IX.—The distribution of the induced currents for an aperiodic primary field, of the type found in magnetic storm-time variations.

Time (hours).	1	2	3	6	9	12	15	18	21	24	30	36	42	48	64	96
								Surface Magnetic Field.								
Primary (e_1)	5·99	10·82	14·72	22·24	25·72	26·93	26·86	26·15	25·09	23·87	21·34	18·93	16·74	14·80	10·63	5·48
Induced (i_1)	2·38	4·09	5·34	7·26	7·61	7·22	6·50	5·68	4·86	4·09	2·75	1·68	0·85	0·20	— 0·84	— 1·44
								Induced Current Intensity.								
I (1·0)	0·4068	0·4922	0·5182	0·4718	0·3768	0·2855	0·2094	0·1496	0·1028	0·0665	0·0165	— 0·0140	— 0·0330	— 0·0453	— 0·0577	— 0·0529
I (0·9)	0·0352	0·1119	0·1786	0·2818	0·2952	0·2700	0·2315	0·1912	0·1539	0·1210	0·0684	0·0323	— 0·0036	— 0·0122	— 0·0377	— 0·0465
I (0·8)	0·0005	0·0098	0·0324	0·1162	0·1716	0·1933	0·2003	0·1789	0·1599	0·1390	0·0984	0·0642	0·0370	0·0158	— 0·0186	— 0·0389
I (0·7)	0·0001	0·0004	0·0031	0·0338	0·0773	0·1113	0·1306	0·1374	0·1357	0·1284	0·1056	0·0800	0·0563	0·0359	— 0·0021	— 0·0310
I (0·6)	0·0000	0·0000	0·0002	0·0072	0·0274	0·0528	0·0749	0·0904	0·0993	0·1022	0·0963	0·0817	0·0642	0·0475	0·0102	— 0·0233
I (0·5)	0·0000	0·0000	0·0000	0·0016	0·0076	0·0208	0·0367	0·0522	0·0643	0·0724	0·0781	0·0733	0·0628	0·0509	0·0178	— 0·0166

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layers increases notably with time ; *cf.* the distribution (fig. 9) at 30h, when the induced field (fig. 8) is still considerable.

The maximum current intensity at each level, relative to the maximum intensity at the surface of the core, in the case of the storm-time field, is shown in fig. 6 by the dotted line ; the deeper penetration of the aperiodic current system, compared with that of the daily periodic variations, is well shown.

Further, figs. 7 and 8 indicate that, at the times when the induced fields at the earth's surface, for the aperiodic harmonic P_1 and the periodic harmonic P_3^2, are at their maximum, the current intensities are distributed in depth as follows :—

Depth from surface of core	0	$0 \cdot 1qa$	$0 \cdot 2qa$	$0 \cdot 3qa$
P_1 (aperiodic)	1	0·77	0·43	0·05
P_3^2 (periodic)	1	0·40	0·0	—0·01

7.7. It is of interest to determine the numerical value of the maximum current-density due to the storm-time variations. This occurs at the surface of the core, in the equatorial plane, about three hours after the main phase begins, and about ten hours before the latter attains its maximum.

The maximum value of $I(\rho)$ is $0 \cdot 52$, at $\rho = 1$; substituting this in (64), and also inserting the values of θ $(= \frac{1}{2}\pi)$ and qa, the current intensity is found to be

$$3.10^{13} \text{ e.m.u.}$$

8. The Electrical Conductivity of the Earth.

8.1. The calculations of §§ 6, 7, have been made on the basis of a uniformly conducting core of radius $0 \cdot 96\, a$ and conductivity $3 \cdot 65.10^{-13}$. These values are in accord with the field of the diurnal magnetic variation, but the storm-time field has been seen (§ 5) to require a higher conductivity (44.10^{-13}) for a core of almost exactly the same size. The discrepancy indicates that the core cannot be uniformly conducting.

The calculations of §§ 6, 7, indicate that, in the assumed core, the current-system associated with the storm-time field penetrates deeper than that of the daily magnetic variations. This is a general result which would be found whatever the assumed properties of the core, because the storm-time variations are aperiodic and slower than the daily periodic variations. If the current-system which penetrates more deeply indicates a higher value for κ, assuming a uniform core of definite size, then it may be inferred that κ increases downwards.

The phase-difference between the primary and induced parts of the field of the daily magnetic variations determines the order of magnitude of κ at the level where the induced field is mainly produced. The amplitude-ratio then fixes the depth of this level. In § 6 it appeared that the induced field was mainly produced between the surface of the core ($r = 0 \cdot 96a$) and the level $r = 0 \cdot 80a$. The calculated value of κ, $3 \cdot 65.10^{-13}$, may be taken to refer to a level of about $0 \cdot 95a$, since the current system,

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particularly for the higher harmonics, is most intense near the surface of the core. If κ is increasing downwards, so that at $0 \cdot 96a$ it is slightly less than $3 \cdot 65.10^{-13}$, and decidedly greater than this below $0 \cdot 95a$, the thickness of the effective part of the current sheet will be less than that illustrated by Table VII.

The induced storm-time field corresponds to a uniform core in which $\kappa = 44 \,.\, 10^{-13}$ below $0 \cdot 94a$, but if κ does not rise beyond $3 \cdot 6 \,.\, 10^{-13}$ till about $0 \cdot 95a$, the excess conductivity required to explain the relatively large induced field must *exceed* $44 \,.\, 10^{-13}$ at the level where the main induced currents are situated. This level is not below $0 \cdot 80a$, according to § 7 and fig. 9, in the first day of a storm; but as the core appears

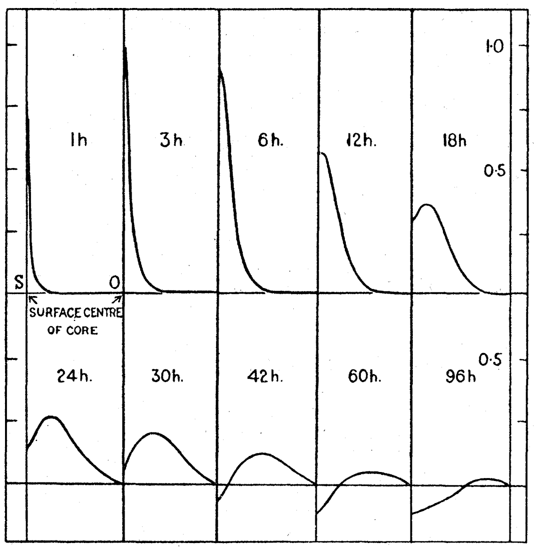

Fig. 9.

to be much more conducting than there assumed, the level must be above this. Without exact calculations of the induction in a sphere of variable conductivity, such as we propose to make, it is not possible to form satisfactory estimates of κ at levels of $0 \cdot 9a$ or below; but it is clear that the magnetic evidence indicates a rather rapid rate of increase of κ with depth between $0 \cdot 96a$ and $0 \cdot 90a$. The slow decrease of the induced currents in the days succeeding a magnetic storm may be ascribed to the passage of the current-pulse (fig. 9) into deeper regions of higher and higher conductivity.

It is interesting to speculate on whether the downward increase of κ is due to some fundamental property of matter when subjected to the high temperatures and intense pressures experienced at these depths.*

9. Electromagnetic Induction by Currents in the Auroral Zones.

9.1. Chapman and Whitehead (*loc. cit.*) have considered the currents and magnetic field induced during the setting up of electric currents in the auroral zones. Such

* *Cf.* the Guthrie lecture by P. W. Bridgman on the "Effects of pressure on the properties of matter," 'Proc. Phys. Soc.,' 41, 341, 1929.

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currents, of the order of magnitude 10^6 amperes, are known to exist in these zones during magnetically disturbed periods. Though their direct magnetic field is considerable in auroral latitudes, it rapidly decreases towards low latitudes, and in the tropics is practically negligible. Their calculations led to the conclusion that the induced field, however, may be of importance in low latitudes, being composed mainly of the harmonic P_1, and therefore increasing towards the equator. This conclusion is erroneous, and was due to an error in the analysis ; the terms representing the complementary function, or "free motion," in the induced currents, were incorrectly chosen, so that the condition of no initial current in the core, at all depths, was not satisfied in their solution. The problem, imperfectly treated by them, is very complex, and an accurate solution is almost impracticable. It may be shown, however, that if the inducing field is negligible in low latitudes, the induced field must also be small.

They showed that westerly currents of amount i (in e.m.u.), in each of the two auroral zones, would together produce a magnetic field of which the potential, near the earth's surface, is

$$i\sum_{n=1}^{\infty} (Z_n r^n/a^{n-1}) P_n (\cos \theta)$$

where the Z's are known numerical coefficients (*cf.* their Table II).

If the zonal currents vary, there will be, in addition to the direct magnetic field of these currents, an induced magnetic field, of which the potential (*cf.* (22)) is

$$\sum_{n=1}^{\infty} (\psi_n a^{n+2}/r^{n+1}) P_n (\cos \theta).$$

From § 3.6, it appears that, whatever the values of κ and μ, the time factor ψ_n (for the period during which i is varying) can only in very exceptional circumstances exceed $iZ_n nq^{2n+1}/(n+1)$, and, even then, only by a small quantity, while, in general, it will be less than this. Further, in their memoir it was found that Z_1 was so small that the contribution of the corresponding first harmonic term to the direct field of the zonal currents was negligible. It follows that the first harmonic term in the induced field will also be negligible.

While the zonal currents are varying, the distribution of the "internal" part (H_i) of the horizontal component of the surface field will be roughly similar to that of the external part (H_e), except that the higher harmonics contributing to H_i will probably be increased slightly in relative importance, owing to the factor $n/(n+1)$ in ψ_n. The external part, H_e, *i.e.*, the horizontal component of the direct field of the zonal currents, has negligibly small values for latitudes less than 50° (*cf.* their Table III) ; the same must therefore be true of H_i. Hence we must conclude that, contrary to their original conclusion, varying electric currents, in circuits approximately coinciding with the auroral zones, cannot give rise, by electromagnetic induction within the earth, to appreciable magnetic fields in low latitudes.

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Computations have also been made of the induced magnetic field just inside a conducting spherical shell, taken to represent a conducting layer in the upper atmosphere. These computations were originally undertaken to test the conjecture in their § 4. In view, however, of the above conclusions, it is unnecessary to give the detailed results of these computations: as above, it appeared that the induced magnetic field would be inappreciable in low latitudes.

It is thus clear that the storm-time variations of the magnetic field in low latitudes cannot be due to currents, induced either in the earth or in a conducting layer of the atmosphere, by varying primary currents in the auroral zones.

10. Note on Maris and Hulburt's Theory of Magnetic Storms.

In a recent paper Maris and Hulburt[13] have formulated a new theory of terrestrial magnetic storms, which they ascribe to the action of ultra-violet solar radiation. In developing that portion of their theory which relates to the storm-time variations of the disturbance field, they attribute important effects to electromagnetic induction within the earth. They regard the first phase of an average storm, during which H is rapidly increased, as due to an outer-atmospheric drift-current circulating along parallels of latitude; this drift-current would give rise to a surface magnetic field, in which H is increased. The authors suppose, however, that the *observed* increase in H is less than that which would be caused directly by the drift currents, in the ratio 1 : 10 (" $10^2\ \gamma$ instead of $10^3\ \gamma$ ") owing to the " opposing current induced in the earth."

But it has been shown in § 3.6 that an increase in the " external " part of H will *always* involve an initial increase in the " internal " (induced) part of H, no matter what values are assumed for the radius, electrical conductivity or magnetic permeability of the earth-core. Moreover, the induced part of H will always be less than the part due to the direct field of the drift-currents. For example, if this direct field is approximately represented by a first zonal harmonic, the part of H due to induction will be approximately one-half the primary part. Hence, instead of the initial rise in H being decreased by induction effects in the ratio 1 : 10, it would be increased in the ratio 3 : 2 approximately; this is in accordance with the spherical harmonic analysis, in § 2, of the actually observed field, into internal and external parts.

With regard to the main phase of an average storm, during which H decreases by about three times as much as its earlier increase, Maris and Hulburt state: " When the atmospheric pulse dies away, which may require several hours, the earth current finally reverses the storm-field to negative values, which decrease to zero as the earth current in turn dies away. The exact calculation of the induced effects leads far into eddy current theory; rough calculation indicates a reasonable earth resistance and damping constant."

The considerations of § 3.6 show that the induced field cannot be of this nature;

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for after the primary field has decayed, the induced field will in general be small, and will certainly never exceed in magnitude the maximum value of the primary field.

11. *Note added on July* 10, 1930.—In an interesting paper* on "Earth Movements and Terrestrial-Magnetic Variations," by Dr. R. GUNN (just received, shortly before the final completion of the present paper), the author refers, among other topics, to the electric conductivity within the earth.

He expresses the view, already considered by CHAPMAN,[11] that the conductivity increases downwards, and points out that the evidence of the daily magnetic variations cannot indicate what is the conductivity at great depths. He goes further, in suggesting rough values for the central conductivity: "the mean temperature of the core is some thousands of degrees, almost certainly more than 3000° and perhaps as high as 10,000°," and at these temperatures the materials must be "highly ionised, and the resistivity must be several orders of magnitude less than that calculated for the surface layers." "No reliable estimate of the resistivity is yet possible, even though several lines of evidence point to values lying between 10^8 and 10^4 e.m.u." We are in general accord with the trend, though not necessarily with the details, of these statements. But the only definite observational evidence at present, and so far as we know, for a downward increase of κ, is that afforded by the present paper and the earlier ones already alluded to. As Dr. Gunn states, magnetic variations of period over a thousand years would be necessary to calculate κ exactly, at great depths, so that much further progress along these lines seems impossible; we can only hope for an advance by progress in the general theory of matter under high pressures and at high temperatures.

Two points may be noted in which Dr. GUNN's paper seems to require correction. On p. 226, he calculates by a rough method, using only the phase-lag between the external and induced field of the daily magnetic variations, that the daily induced currents flow in a layer about 200 km. thick, extending downwards from the surface, of resistivity about $1 \cdot 4.10^{12}$ e.m.u., which he regards as consistent with the earth-resistivity measures made by Messrs. GISH and ROONEY. In so doing, however, he ignores the amplitude-ratio of the two fields, which shows that the major part of the currents flow *below* a depth of about 200 km. This is in itself an indication that below the depths accessible by the means used in the valuable work of GISH and ROONEY, *i.e.*, below a few km., the resistivity is distinctly higher than their lowest values (about 10^{13}).

Further, on p. 228, he suggests that the lunar daily magnetic variation is capable of qualitative explanation by body tides in the earth; but spherical harmonic analysis has shown[3] quite definitely that this variation is of external origin, and the internal part is fully explicable as due to currents induced in the earth by the primary outer field.

* 'Transactions of the American Geophysical Union,' 1929 and 1930, published by the National Research Council, June, 1930; pp. 225–228. [This paper has since been reprinted in Terr. Mag., vol. 35, p. 151 (1930).]